A short summary of biophysics

Peter Latham, January 29, 2026

1 Introduction

The brain consists of a large number of neurons – about 100 billion in humans – that mainly communicate via spikes. (The "mainly" qualifier is because neurons can directly communicate their voltage via gap junctions. Here we ignore gap junctions, as they don't play much of a role in mature mammals.) The communication is kind of complicated, as neurons consist of multiple parts: a soma (cell body), as well as dendrites, axons and synapses (Fig. 1).

The goal here is to understand how neurons communicate via spikes. We'll do this in stages: we'll first consider the soma, then dendrites and axons, and, finally, the synapses. We'll start, though, with a brief introduction to biophysics in general.

2 Biophysics

In biophysics the main thing we're interested in is the membrane potential, $V(t)$, which is the voltage difference between the inside and outside of a neuron. As shown in Fig. 1, neurons have three main parts: soma, dendrites, and axons, and the membrane potential

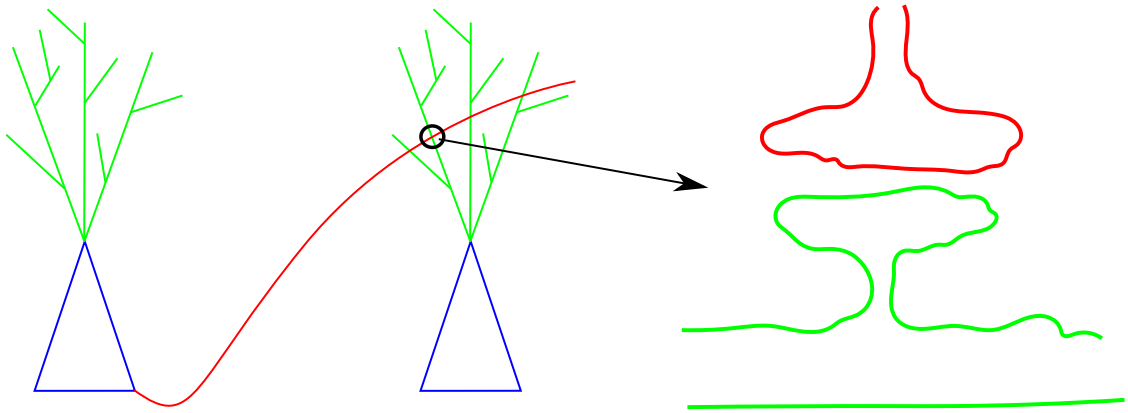

Figure 1: Coupled neurons. The two objects on the left are neurons (which don't really all look alike; I was just too lazy to make them different). The neurons have three main parts: soma (blue), dendrites (green) and axons (red). The dendrites are much, much bigger than shown (50-100 times the size of the soma, which is on the order of 10-20 microns), and so are the axons, which branch (because they connect to about 1,000 other neurons), and can travel long distances (up to a meter). Neurons communicate via synapses, which connect axons to dendrites (usually; axons can also connect directly to the soma). A typical synapse is shown on the right: the presynaptic terminal (red) connects to a spine (green), which is a small structure that sticks out of the dendrites. This being biology, a spine is not always present; the conventional wisdom is that excitatory neurons connect to spines and inhibitory neurons connect directly to dendrites or to the soma. But, this being biology, that conventional wisdom is often violated.

isn't the same everywhere. Ultimately we have to derive separate equations for the membrane potential on the different parts of the neurons. But in this section we'll just think of $V(t)$ as the voltage difference between the inside and outside of a membrane. Given that, we'll derive a general equation for its time derivative.

Essentially, we use two equations. The first is $Q = CV$ where V is voltage and Q is the net charge on the inside of the membrane. Taking a time derivative (and noting that $dQ/dt =$ current) gives us

$$C\,\frac{dV}{dt} = -I\,. \tag{1}$$

Here I is, by convention, the outward current – the current flowing from inside to outside. The sign should make sense: if I is positive, current flows out and the voltage goes down; if I is negative, current flows in and the voltage goes up.

Equation (1) is absolutely fundamental. OK, sort of absolutely fundamental: it ignores magnetic fields, and assumes that the voltage is the same everywhere inside the membrane, which isn't always the case (in particular, it's not the case for dendrites and axons, but we have ways of dealing with that). For now, though, we'll assume that Eq. (1) holds.

So what's the current? If charge were carried by electrons, the current would be computed from $V = IR$ where R is resistance, and if R were constant, we would have a classic RC circuit,

$$C\,\frac{dV}{dt} = -V/R \tag{2}$$

which has the solution $V(t) = V(0)e^{-t/RC}$. However, neurons are not nearly this simple, so the equations are a bit more complicated. For several reasons.

First, charge is not carried by electrons, it's carried by ions. And, because neurons have ion pumps, the ions have different concentrations on the inside and outside of the cell. In particular, the concentrations of sodium and chloride (abbreviated Na and Cl) are high on the outside of the cell, while the concentration of potassium (abbreviated K) is high on the inside. (If you ever become a neuroscientist you should memorize that; but if not I wouldn't bother; it's one of those facts you can always look up.) What's important is the effect of an ion imbalance: even when the membrane potential, V, is zero, an ion imbalance will cause a current to flow (for example, an inward Na current, because there's a lot more sodium on the outside than inside). That rules out $V = IR$, and it means we need something more complicated. The thing we use is

$$I_x = g_x(V - \mathcal{E}_x) \tag{3}$$

where x refers to the ion, so it could be Na, Cl or K (other common ions used in the brain are Ca, for calcium, and Mg, for magnesium, but we won't worry about either, at least for now). The parameter g_x is the conductance of a channel that allows ion x to pass through (it's the inverse of the resistance, R_x: $g_x = 1/R_x$), and $\mathcal{E}_x$ is the reversal potential. The reversal potential needs to be included because of the concentration imbalance. For example, the reversal potential for Na is about 20 mV, which means the voltage on the inside of the cell has to be about 20 mV higher than the voltage on the outside to keep the sodium current from flowing.

Notice that the conductance depends on the ion. That's because channels, which are holes in the cell that ions can flow through, can be ion specific. For example, a channel

may allow only Na, or only Cl, to flow through it. But because this is biology, which is inherently complicated, some channels aren't ion specific, and they let any ion flow through them (although often with different conductances). And, of course, there's the in-between case: channels that let a few ions through, like Na and K but nothing else. But that doesn't really matter; g_x measures the total conductance of ion x taken over the whole membrane.

A nice thing about conductances is that they add, which should be kind of intuitive: adding more channels gives you more current (remember parallel circuits?). Thus, the total current is

$$I = \sum_x g_x(V - \mathcal{E}_x)\,. \tag{4}$$

It is useful to combine Eqs. (1) and (4), which gives us

$$C\,\frac{dV}{dt} = -\sum_x g_x(V - \mathcal{E}_x)\,. \tag{5}$$

This is the starting point for pretty much all of biophysics, and is the main equation we'll use.

The second thing that makes the equations more complicated (and more interesting!) is that the conductances, g_x can depend on just about anything. In the simplest case, they're constant, which gives us a passive neuron. Passive neurons are simple, but not very useful as computing devices. Consequently, evolution invented voltage-dependent conductances (to generate spikes) and concentration-dependent conductances (to allow communication across synapses). We'll consider those below, and we'll also examine how all this changes for extended objects like dendrites and axons.

3 The Hodgkin-Huxley model

Here we consider active channels. Channels themselves are very small, consequently, they're stochastic. Typically we model them as being either open or closed. Importantly, the probability of opening or closing depends on voltage, which we write

$$\begin{aligned} \alpha_x(V) &= \text{probability per unit time that channel } x \text{ goes from closed to open} \\ \beta_x(V) &= \text{probability per unit time that channel } x \text{ goes from open to closed.} \end{aligned} \tag{6}$$

where x refers to channel type. This is a Markov model, so if x is the probability that the channel is open, then it is is (relatively) easy to show that x obeys the equation

$$\tau_x(V)\,\frac{dx}{dt} = x_\infty(V) - x \tag{7}$$

where

$$\tau_x(V) = \frac{1}{\alpha_x(V) + \beta_x(V)} \tag{8a}$$

$$x_\infty(V) = \frac{\alpha_x(V)}{\alpha_x(V) + \beta_x(V)}\,. \tag{8b}$$

OK, this isn't quite the whole story; a channel typically consists of more than one x, and they all have to be open for current to flow. For the Hodgin-Huxley model, there are two kinds of active channels: sodium and potassium. The probability that the active sodium channel is open is m^3h and the probability that the active potassium channel is open is n^4, where m, h and n obey Eq. (7), but with x replaced with the appropriate variable. For this model, the voltage evolves according to

$$C\,\frac{dV}{dt} = -g_L(V-\mathcal{E}_L) - g_{Na}m^3h(V-\mathcal{E}_{Na}) - g_K n^4(V-\mathcal{E}_K) \tag{9}$$

where g_L is the leak (meaning passive) conductance and m, h and n are the probability of channels being open. We typically divide by g_L to give us the equation

$$\tau\,\frac{dV}{dt} = -(V-\mathcal{E}_L) - \rho_{Na}m^3h(V-\mathcal{E}_{Na}) - \rho_K n^4(V-\mathcal{E}_K) \tag{10}$$

where

$$\tau = \frac{C}{g_L} \approx 10\text{ms} \tag{11a}$$

$$\rho_{Na} = \frac{g_{Na}}{g_L} \approx 400 \tag{11b}$$

$$\rho_K = \frac{g_K}{g_L} \approx 240\,. \tag{11c}$$

To understand the behavior of these equations, recall first of all that $\mathcal{E}_{Na} \approx +20$ mV and $\mathcal{E}_K \approx -80$ mV. Second, we need to know how $m_\infty(V)$, $h_\infty(V)$, and $n_\infty(V)$, The first and last, m and n, are increasing functions of V, while h is a decreasing function of V. Thus, when the voltage increases past a threshold, the m-channels open, which raises the voltage even more, which causes them to open even more. This leads to a rapid increase in voltage. However, with a slight delay, the h-channel closes, pushing the voltage toward the leak potential, and at the same time the n-channel opens, pushing the voltage down even more, toward -80 Mv. For this to work, the time constant of the m-channel must be much smaller than for the other two, which is it: $\tau_m < 1$ ms, while τ_h and τ_n are both on the order of 1-2 ms.

We would like to make this quantitative picture more quantitative, but that's hard – the Hodgkin-Huxley model is a nonlinear differential equation with four variables, and as far as anybody knows there's no analytic solution. So we do what any self-respecting theorist does: we change the problem. Because the m-channel is fast, we replace m in Eq. (10) with $m_\infty(V)$. And because the potassium channel just causes the voltage to undershoot, we get rid of it altogether. This gives us the two-variable system

$$\tau\,\frac{dV}{dt} = -(V-\mathcal{E}_L) - \rho_{Na}m_\infty(V)^3h(V-\mathcal{E}_{Na}) + V_0\,. \tag{12a}$$

$$\tau_h(V)\,\frac{dh}{dt} = h_\infty(V) - h\,. \tag{12b}$$

Note that we have added an external voltage (which is really an external current times some conductance), because we may want to drive the neuron. To analyze these equations we draw the nullclines: curves along which either $dV/dt = 0$ (the V-nullcline) or $dh/dt = 0$ (the h-nullcline). These are shown in Fig. 2. For the reversal potentials I used $\mathcal{E}_L = -70$ mV

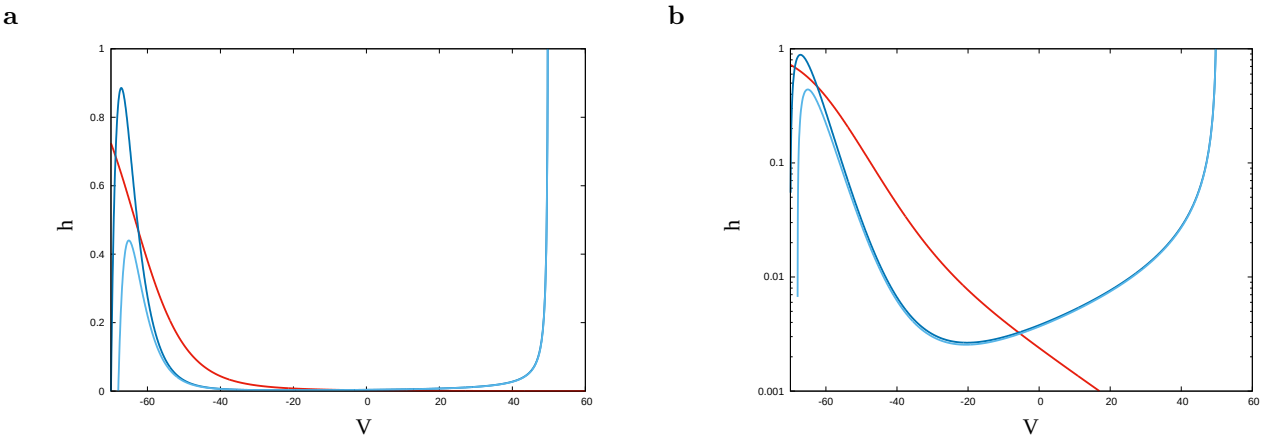

Figure 2: Nullclines for the approximate model given in Eq. (12). Red: h-nullcline. Dark blue (higher curve): V-nullcline with $V_0 = 0$. Light blue (lower curve): V-nullcline with $V_0 = -2$.

and $\mathcal{E}_{Na} = 50$ mV, and for the ratio of peak sodium to leak conductance I used $\rho_{Na} = 400$. Finally, for $h_\infty(V)$ and $m_\infty(V)$ I used,

$$h_\infty(V) = \frac{0.06e^{-0.05(V+65)}}{0.06e^{-0.05(V+65)} + 1/(1+e^{-0.1(V+35)})} \tag{13a}$$

$$m_\infty(V) = \frac{0.1(V+40)(1-e^{-0.1(V+40)})}{0.1(V+40)(1-e^{-0.1(V+40)}) + 4e^{-0.0556(V+65)}}\,, \tag{13b}$$

taken from Dayan and Abbott (I think). Because this is a 2-D picture, dynamics and stability is relatively easy to analyze.

4 Dendrites and axons

For dendrites and axons, we can no longer assume that $V(t)$ is constant everywhere on the inside of the membrane. Which makes things a bit more complicated. To deal with this case, we'll derive the famous passive cable equation, which describes the propagation of voltage in dendrites. Then we'll consider axons, which are basically dendrites with a bit of insulation and some active channels.

4.1 The cable equation

We'll initially assume that a dendrite is an infinitely long cylinder. A short section of a dendrite is shown in Fig. 3 (see the figure caption for details). We'll use, as usual, $V = IR$ and $Q = CV$, the latter implying $CdV/dt = I$. We'll start with $CdV/dt = I$. Treating each section as equipotential, we have

$$C_m \frac{\partial V(x,t)}{\partial t} = I_L(x - dx/2, t) - I_L(x + dx/2, t) - I_m(x,t) + I_e(x,t) \tag{14}$$

where I_L is the longitudinal current, I_m is the current due to membrane channels, I_e is the injected current, C_m is the membrane capacitance and R_L is the longitudinal resistance. And

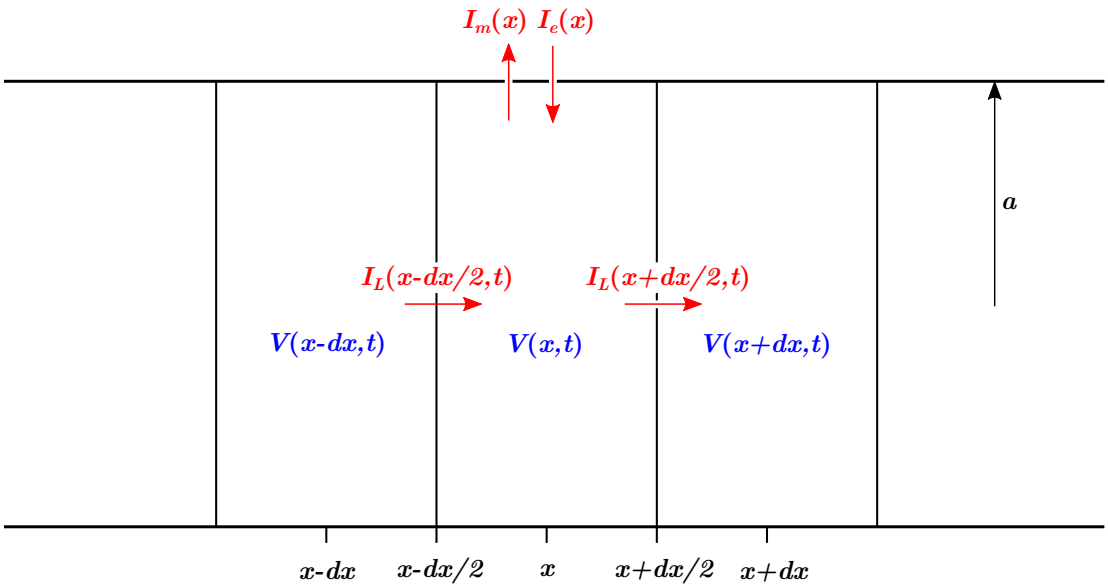

Figure 3: A short section of a cylindrical dendrite with radius a. The vertical bars are spaced by dx, which we'll eventually take to zero, so the drawing is slightly misleading: the bars should be very close together (think stacks of pancakes), but then there would be no room to show the currents. There are three kinds of currents: longitudinal current, I_L, which flows along the dendrite; current due to membrane channels, I_m (which is outward by convention); and injected current, I_e (which is inward by convention).

now we use $V = IR$,

$$I(x - dx/2, t) = \frac{V(x - dx, t) - V(x)}{R_L} \tag{15}$$

where R_L is the longitudinal resistance; a similar equation applies to $I(x + dx/2, t)$. (Note that we're using partial derivatives than total derivatives; that's because we'll soon be taking derivatives with respect to x as well.) Inserting Eq. (15) into (14), we have

$$C_m \frac{\partial V(x,t)}{\partial t} = \frac{V(x - dx, t) - V(x, t)}{R_L} - \frac{V(x, t) - V(x + dx, t)}{R_L} - I_m(x, t) + I_e(x, t). \tag{16}$$

Taylor expanding membrane potential to second order in dx gives us

$$C_m \frac{\partial V(x,t)}{\partial t} = \frac{dx^2}{R_L} \frac{\partial^2 V(x,t)}{\partial x^2} - I_m(x, t) + I_e(x, t). \tag{17}$$

That's the easy part. The hard(er) part is taking the limit $dx \to 0$. Basically, we have to figure out how C_m and R_L scale with dx. We'll start with R_L. It's a physics fact that resistance scales linearly with the length of a material and inversely with its area, with a scale factor that's an intrinsic property of the material. For the longitudinal resistance, we'll use r_L for that intrinsic property, giving us

$$R_L = r_L \frac{dx}{\pi a^2}\,. \tag{18}$$

Here dx is the length and πa^2 is the area (remember the dendrite is cylindrical).

Capacitance also depends on area. To see how, we need a physics fact about membrane potential: inside the dendrite, at any particular value of x, the membrane potential is constant. That means $V(x)$ is the change in potential across the membrane, and that change is caused by charge building up on the inside and outside of the membrane. It follows (with a little thinking) that for fixed membrane potential the total charge scales linearly with area, with a scale factor the depends on the membrane. We'll use c_m for that scale factor, giving us

$$C_m = c_m 2\pi a dx. \tag{19}$$

Inserting Eqs. (18) and (19) into Eq. (17), and performing a small amount of algebra, we arrive at

$$c_m \frac{\partial V(x,t)}{\partial t} = \frac{a}{2r_L} \frac{\partial^2 V(x,t)}{\partial x^2} - i_m(x,t) + i_e(x,t) \tag{20}$$

where i_m and i_e are current densities,

$$i_m(x,t) \equiv \frac{I_m(x,t)}{2\pi a dx} \tag{21a}$$

$$i_e(x,t) \equiv \frac{I_e(x,t)}{2\pi a dx}\,. \tag{21b}$$

Equation (20) is independent of dx, as desired.

For dendrites (but, as we'll see in the next section, not for axons), it's convenient to do one more thing. Equation (18) is a general expression that relates total resistance to geometry and intrinsic properties. For something with fixed thickness – like the membrane of a neuron or dendrite – we can think of $r_L \times$thickness as the intrinsic membrane property, which we'll called r_m. This gives us

$$R_m = \frac{r_m}{\text{Area}} \tag{22}$$

where R_m is the actual resistance. We're going to multiply both sides of Eq. (20) by r_m. This gives us a factor of $r_m c_m$ on the left hand side. To interpret this factor, we note, as discussed above, that capacitance is proportional to area. And so we may write

$$R_m C_m = \left(\frac{r_m}{\text{Area}}\right)\left(c_m \text{Area}\right) = r_m c_m \tag{23}$$

where C_m is the total capacitance. In general, $R_m C_m$ is the membrane time constant; thus, so is $r_m c_m$. If dendrites have similar properties to cell somas, which, it turns out, they do, then $r_m c_m$ is on the order of 10 ms.

Multiplying both sides of Eq. (20) by r_m, and defining

$$\tau_m \equiv r_m c_m, \tag{24}$$

we have

$$\tau_m \frac{\partial V(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 V(x,t)}{\partial x^2} - r_m i_m(x,t) + r_m i_e(x,t) \tag{25}$$

where λ, which is known as the electrotonic length, is given by

$$\lambda^2 \equiv \frac{a r_m}{2 r_L}\,. \tag{26}$$

Equation (25) is the cable equation. In real dendrites, the current density, i_m, consists of both passive and active channels (the active channels are just like the ones we saw in the Hodgkin Huxley equations). Here, however, we'll consider only passive channels,

$$I_m = \frac{V-\mathcal{E}}{R_m} = \frac{2\pi a dx}{r_m}(V-\mathcal{E}); \tag{27}$$

comparing this to Eq. (21a), we see that

$$r_m i_m = V - \mathcal{E}, \tag{28}$$

yielding

$$\tau_m \frac{\partial V(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 V(x,t)}{\partial x^2} - (V(x,t)-\mathcal{E}) + r_m i_e(x,t). \tag{29}$$

This is the famous passive cable equation. We're going to solve it first in steady state ($V(x,t)$ is independent of time); then we'll tackle the time-dependent case. In either case, we can simplify our equations by working with voltage relative to $\mathcal{E}$; we do that by defining a new variable,

$$u \equiv V - \mathcal{E}, \tag{30}$$

giving us the equation

$$\tau_m \frac{\partial u(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + r_m i_e(x,t). \tag{31}$$

4.2 The response of a dendrite to steady injected current

To compute the membrane potential in steady state, we need to specify the current density, $i_e(x,t)$. We'll assume that the current is injected at one point, so it's something like $I_0/2\pi a dx$ for x within $dx/2$ of zero and zero otherwise. In the limit that $dx \to 0$, the factor of $1/dx$ turns into a delta-function, leading to the equation

$$\tau_m \frac{\partial u(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} - u + \frac{r_m I_0}{2\pi a}\delta(x) \tag{32}$$

where $\delta(x)$ is the Dirac delta-function; it has the property that it is zero when $x \neq 0$ and it integrates to 1: so long as a and b are positive, then

$$\int_{-a}^{b} dx\, \delta(x) = 1. \tag{33}$$

The delta-function is one of the most useful functions in the world; it's a good idea to know about it.

If we let the cable equation evolve for a long time, eventually we'll reach steady state – meaning $\partial u(x,t)/\partial t = 0$. Thus, in the limit $t \to \infty$, our cable equation becomes

$$\lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} - u = -R_\lambda I_0 \lambda \delta(x) \tag{34}$$

where

$$R_\lambda \equiv \frac{r_m}{2\pi a \lambda} = \frac{r_L \lambda}{\pi a^2}\,. \tag{35}$$

As an aside, R_λ has a very natural interpretation: Recall that total resistance of a material is equal to resistivity, r_L times the length of the material divided by its area. Thus, R_λ is the resistance of a dendrite with radius a and length λ. Which, in hind sight, is not especially surprising. But it's kind of cool.

Solving equations with delta-functions is often easy, and that's the case here. That's because when $x \neq 0$, we have a simple linear ODE. The only problem is to figure out what happens at $x = 0$. For that we just have to take derivatives of discontinuous functions. If you know that, or you don't know it and don't care, you can skip the next couple of paragraphs. However, you'll need it to understand the solution to Eq. (34).

Consider a function $g(x)$ that's continuous everywhere except at $x = x_0$. For instance, we might have $g(x) = x^2$ if $x < x_0$ and $g(x) = x^2 + 4$ if $x > x_0$. We'll define g_0 and g_1 to be the values of $g(x)$ when approached from the below and above x_0, respectively. In the above example, $g_0 = x_0^2$ and $g_1 = x_0^2 + 4$. To compute a derivative, we'll use the usual expression,

$$\frac{dg(x)}{dx} = \lim_{\epsilon \to 0} \frac{g(x + \epsilon/2) - g(x - \epsilon/2)}{\epsilon}\,. \tag{36}$$

Most of the time this just gives us the derivative. However, if $x = x_0$, things are slightly more complicated,

$$\frac{dg(x)}{dx} = \lim_{\epsilon \to 0} \frac{g_1 - g_0}{\epsilon}\,, \tag{37}$$

which goes to ∞. To figure out how big the infinity is, we note that

$$g(x + a) = g(x - b) + \int_{x-b}^{x+a} dy\, \frac{dg(y)}{dy}\,. \tag{38}$$

If we take $x = x_0$ and both a and b infinitesimally small, the integral should equal $g_1 - g_0$. This is achieved if $dg(y)/dy = (g_1 - g_0)\delta(x)$. Thus, the infinity is equal to the infinity associated with the Dirac delta-function. Derivatives at discontinuities, then, yield delta-functions times the size of the discontinuity,

$$\frac{dg}{dx} = \left.\frac{dg}{dx}\right|_{\text{continuous}} + (g_1 - g_0)\delta(x - x_0) \tag{39}$$

where the subscript "continuous" means the continuous part of the derivative.

Given the above discussion, and the fact that $u(x) \to 0$ when $x \to \pm\infty$, it's not hard to figure out, or at least verify, that Eq. (34) has the solution

$$u(x) = \frac{I_0 R_\lambda}{2}\, e^{-|x|/\lambda}\,. \tag{40}$$

Parameter	Value	Relation to variables in our derivation
r_L	1 kΩ-mm	$R_L = r_L dx/\pi a^2$
r_m	1 MΩ-mm^2	$R_m = r_m/\text{Area}$
c_m	10 nF/mm^2	$C_m = c_m \times \text{Area}$
λ	$\sqrt{r_m a/2r_L}$	electrotonic length

Table 1. Dendritic parameters. All numbers are approximate.

Thus, λ – the electrotonic length – determines the spread of voltage in response to steady injected current.

So how big is λ? Using its definition (Eq. (26)) and the values of r_L and r_m given in Table 1, we have

$$\lambda(\text{mm}) = \sqrt{\frac{a\,(\mu\text{m})}{2}}\,. \tag{41}$$

Given that a is on the order of one μm, the electrotonic length is on the order of 1 mm. This places a fundamental limit on the length of dendrites: if they're much longer than 1 mm, steady input at the distal ends of the dendrites will not be seen by the soma.

Before going to the time dependent case, we want to do one more thing: find $u(x)$ when the input current density, $i_e(x)$, is a smooth function of x (rather than the δ-function considered above). The method for doing that is general, but I'll be honest: I don't use it very much. Still, if you're planning on doing math for a living, it's worth knowing about.

Let's consider a slight modification to Eq. (34): we'll center the delta-function around $x = x'$, and we won't include the multiplicative constants. The resulting equation is

$$\lambda^2 \frac{\partial^2 G(x-x')}{\partial x^2} - G(x-x') = -\delta(x-x'). \tag{42}$$

The function $G(x-x')$ is called the Green function, presumably named after Green. It's easy to solve this equation: using our previous solution (Eq. (40)), but shifted by x', we have

$$G(x-x') = \frac{1}{2\lambda}\, e^{-|x-x'|/\lambda}\,. \tag{43}$$

Now consider the function

$$u(x) = \int dx'\, G(x-x') r_m i_e(x')\,. \tag{44}$$

Using Eq. (42) for the Green function, we see that

$$\lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} - u = -\int dx'\, \delta(x-x') r_m i_e(x') = -r_m i_e(x). \tag{45}$$

Thus, if we can compute the Green function, we can find the steady state solution to Eq. (31) for any time independent current density $i_e(x)$ just by performing a convolution! That's often very convenient, and it's used extensively in quantum field theory.

4.3 The response of a dendrite to a time-dependent injected current

Given our experience with the Green function above, it's enough to know the solution for a delta-function current – so long as it's a delta-function over time as well as space. We thus consider the equation

$$\tau_m \frac{\partial u(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,y) + I_0 R_\lambda \lambda \tau_m \delta(x)\delta(t). \tag{46}$$

The constants in front of the delta-functions are there to make the units work out. Once we solve this equation, it will be easy to solve the equation with the delta-functions at arbitrary points in space and time, $\delta(x)\delta(t) \to \delta(x-x')\delta(t-t')$, and then easy to fund $u(x,t)$ in response to an arbitrary current density $i_e(x,t)$.

There are several ways to solve Eq. (46), probably the easiest of which is to Fourier transform with respect to x, solve the resulting ordinary differential equation in t, and then Fourier transform back. But I won't go into detail; instead I'll just write down the solution,

$$u(x,t) = \frac{I_0 R_\lambda}{\sqrt{4\pi(t/\tau_m)}} \exp\left[-\frac{(x/\lambda)^2}{4(t/\tau_m)}\right] e^{-(t/\tau_m)}\,\Theta(t) \tag{47}$$

where $\Theta(t)$ is the Heaviside step function: $\Theta(t) = 1$ if $t > 0$ and 0 otherwise. You should verify this is the solution to Eq. (46). That's easy to do for $t > 0$; it's much harder to do when $t = 0$. There's also the problem of verifying the overall normalization, but there's a trick for that: integrate over all x,

$$\tau_m \frac{d}{dt}\int_{-\infty}^{\infty} dx\, u(x,t) = \lambda^2 \int_{-\infty}^{\infty} dx\, \frac{\partial^2 u(x,t)}{\partial x^2} - \int_{-\infty}^{\infty} dx\, u(x,t) + I_0 R_\lambda \lambda \tau_m \delta(t)\,. \tag{48}$$

Assuming $du(x,t)/dx$ vanishes at $\pm\infty$ (an assumption we can make on physical grounds), the first term on the right hand side is zero. consequently,

$$\int_{-\infty}^{\infty} dx\, u(x,t) = I_0 R_\lambda \lambda \Theta(t) e^{-t/\tau_m}\,. \tag{49}$$

This is consistent with the overall normalization in Eq. (47).

Equation (47) tells us that at any point in time, $u(x,t)$ is Gaussian with width proportional to $\lambda\sqrt{t/\tau_m}$. The fact that the width is proportional to λ is consistent with what we found in the steady state case. However, because of the factor of $\sqrt{t/\tau_m}$, it might seem that the membrane potential could spread much farther in the time-dependent case than in the time-independent case. However, this isn't really true: if t is large compared to τ_m, it's true that the Gaussian will be wide compared to λ. But because of the factor $e^{-(t/\tau_m)}$, its amplitude will be exponentially small. Thus, again the spread of voltage can't be much more than λ.

While the x-dependence is easy to understand, it's not all that relevant to the soma. We'll ignore for the moment the fact that dendrites branch, and think of a dendrite as being infinitely long, and place the soma a distance L from the point where the current is injected. In that case, the time dependence of the voltage at the soma is given by

$$u(L,t) \propto \exp\left[-\frac{(L/\lambda)^2}{4(t/\tau_m)} - (t/\tau_m) - \frac{1}{2}\log(t/\tau_m)\right] \tag{50}$$

At small times the voltage is suppressed by the first term in brackets; at large times it's suppressed by the second, and, to a lesser extent, the third term. In between is a maximum, which we can find by differentiating the term in brackets with respect to t and setting the resulting expression to zero. That maximum, which we'll denote $t^*(L)$, is given by

$$\frac{t^*(L)}{\tau_m} = \frac{\sqrt{4(L/\lambda)^2+1}-1}{4}\,. \tag{51}$$

We can define the "speed" of propagation, denoted v, as the ratio of L to $t^*(L)$,

$$v = \frac{L}{t^*(L)} = \frac{\lambda}{\tau_m}\frac{4L/\lambda}{\sqrt{4(L/\lambda)^2+1}-1}\,. \tag{52}$$

When L is small compared to λ, $v \propto \lambda^2/L\tau_m$. Thus, for very short distances the speed is high; that's because the cable equation is a diffusion equation, for which the width grows as $\sqrt{t}$. When L is large, on the other hand, things are much better behaved: $v \to 2\lambda/\tau_m$. Given the definition of λ (Eq. (26)), this implies that the speed scales with the square root of the radius. That's a fact that many people seem to know, but the truth of the matter is that it simply doesn't come up that often.

It's also important to determine the maximum voltage at $x = L$. Inserting Eq. (51) into Eq. (47) gives us

$$u\big(L, t^*(L)\big) = \frac{I_0 R_\lambda}{\sqrt{4\pi}} \exp\left[-\frac{\sqrt{4(L/\lambda)^2+1}}{2} - \frac{1}{2}\log\left(\frac{\sqrt{4(L/\lambda)^2+1}-1}{4}\right)\right]\,. \tag{53}$$

Not surprisingly, for large L this falls off as $e^{-L/\lambda}$. Thus, the signal in dendrites can't propagate any farther when the input current is time time-dependent than when it's static.

4.4 Beyond cylinders: branching

Dendrites are not, of course, simply cylinders. A reasonable approximation is to treat them as cylinders that occasionally branch. If we wanted to solve the cable equations at a branch point, we just need to know the boundary conditions. Those are relatively simple: both the current and voltage are continuous. The current needs to be continuous because otherwise there would be a buildup of charge; the voltage needs to be continuous because the current is proportional to $\partial V(x,t)/\partial x$ (see Eq. (15) and note that $R_L \propto dx$); if there were discontinuities, the current would go to infinity. (We saw this in the above analysis: when the current was proportional to a delta-function (Eq. (34)), there was a discontinuity in the voltage (Eq. (40)).

We'll consider a typical branch point, as shown in Fig. 4, with constant current, I_0, injected a distance y from the branch point. The red arrows indicate both the direction of the current and the direction of the three coordinates, x_1, x_2 and x_3. We'll start by writing down an expression for the voltage in the three branches. For branches 2 and 3 we just have exponential decay away from 0, but for branch 1 we also have the term associated with the injected current. Using the steady state solution given in Eq. (40), we see that

$$V(x_1) = \frac{R_{\lambda_1} I_0}{2}\, e^{-|x_1-y|/\lambda_1} + A_1 e^{-x_1/\lambda_1} \tag{54a}$$

$$V(x_2) = A_2 e^{-x_2/\lambda_2} \tag{54b}$$

$$V(x_3) = A_3 e^{-x_3/\lambda_3} \tag{54c}$$

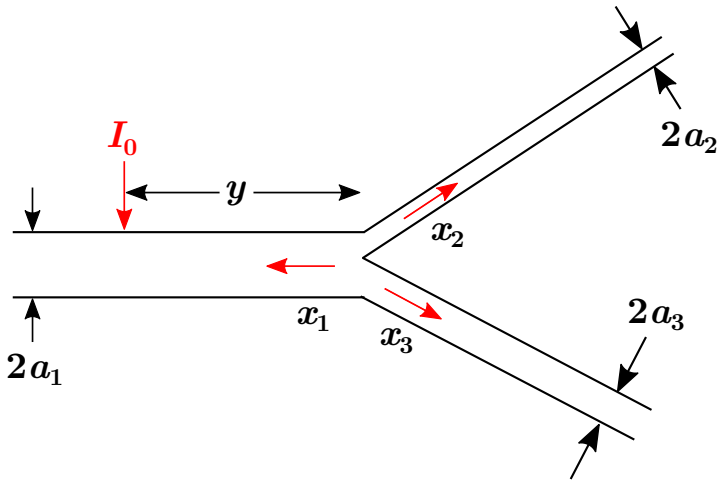

Figure 4: Dendritic branch point. Each branch has a different radius. We adopt a convention in which current is outward for all branches, and distance increases in the outward direction. Constant current is injected a distance y from the branch point in branch 1. Our goal is to find the membrane potential in each branch.

where A_1, A_2 and A_3 are constants that we will determine shortly. For the currents, use take the $dx \to 0$ limit of Eq. (15) and use Eq. (18) for R_Li, which together imply that $I(x) = (\pi a^2/r_L)\partial V(x)/\partial x$; resulting in

$$I(x_1) = -\frac{R_{\lambda_1} I_0}{2} \frac{\pi a_1^2}{r_L \lambda_1} e^{-|x_1 - y|/\lambda_1} \text{sign}(x_1 - y) - \frac{\pi a_1^2}{r_L \lambda_1} A_1 e^{-x_1/\lambda_1} \tag{55a}$$

$$I(x_2) = -\frac{\pi a_2^2}{r_L \lambda_2} A_2 e^{-x_2/\lambda_2} \tag{55b}$$

$$I(x_3) = -\frac{\pi a_3^2}{r_L \lambda_3} A_3 e^{-x_3/\lambda_3} \,. \tag{55c}$$

To find the values of A_1, A_2 and A_3, we use our boundary conditions on voltage and current. The first is that the voltage is continuous at 0, giving us

$$\frac{R_{\lambda_1} I_0}{2} e^{-y/\lambda_1} + A_1 = A_2 = A_3. \tag{56}$$

The second is that the total current, $I_1 + I_2 + I_3$, is zero. Using the fact that $\lambda \propto a^{1/2}$ (see Eq. (26)), zero total current implies that

$$\frac{R_{\lambda_1} I_0}{2} a_1^{3/2} e^{-y/\lambda_1} - a_1^{3/2} A_1 - a_2^{3/2} A_2 - a_3^{3/2} A_3 = 0. \tag{57}$$

Solving the above two equations gives us

$$A_1 = \frac{R_{\lambda_1} I_0}{2} e^{-y/\lambda_1} \frac{a_1^{3/2} - a_2^{3/2} - a_3^{3/2}}{a_1^{3/2} + a_2^{3/2} + a_3^{3/2}} \tag{58a}$$

$$A_2 = A_3 = R_{\lambda_1} I_0 e^{-y/\lambda_1} \frac{a_1^{3/2}}{a_1^{3/2} + a_2^{3/2} + a_3^{3/2}} \,. \tag{58b}$$

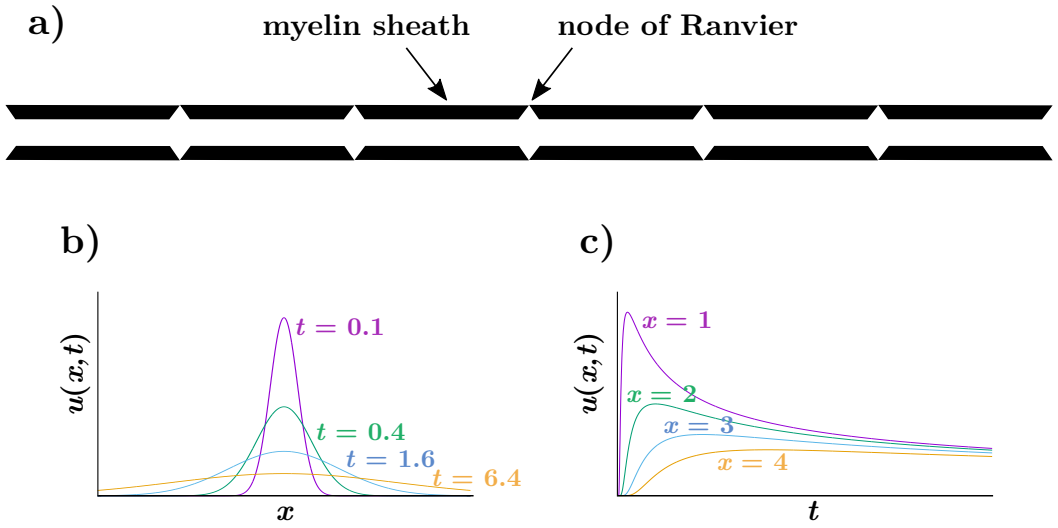

Figure 5: Propagation along axons. **a**. A section of an axon. The dark regions indicate the myelin sheath; the notches are the nodes of Ranvier, which contain active sodium and potassium channels. **b**. Membrane potential relative to the leak reversal potential versus x at various times. **c**. Membrane potential relative to the leak reversal potential versus t at various locations. For both **a** and **b** the diffusion constant, D, was 1.

If $a_1^{3/2} = a_2^{3/2} + a_2^{3/2}$, then $A_1 = 0$. This is the famous "three halves power law". When the radii satisfy this relationship, there is no "reflected" voltage from the branch point. More interestingly, if all the branch points on a dendritic tree satisfy this relationship, a branching dendrite can be replaced by a single "equivalent" cylinder. For details on how to do that, see Dayan and Abbott.

5 Axons

The fact that information can't propagate very far in dendrites means we need a different structure to support the long range propagation that's necessary in animals larger than about 1 mm. That brings us to axons. Axons have two properties that increase the range of signal propagation. First, they have a myelin sheath wrapped around them. That sheath has effectively infinite resistance, so no current flows in or out. Consequently, the time dependent solution that we found in Eq. (47) is missing the decay term, e^{-t/τ_m}. That helps, but not much; the solution is still a spreading Gaussian, the spread is slow at large times, and the amplitude falls off with time.

The second, and more important, property of axons is that they have active channels. These are situated at the nodes of Ranvier – locations along the axon where the myelin sheath is missing (Fig. 5a). Nodes of Ranvier contain active sodium channels – like the Hodgkin neuron, so when the voltage is high enough an action potential occurs. When that happens, there is a very brief, very large, inward current, which causes the membrane potential to quickly climb to about 0 mV.

Propagation of an axon, then, proceeds as follows. Let's start at the soma, and assume it spikes. (It turns out that spike initiation is often in the axon, a few tens of microns from the soma, but that's a detail). When a spike occurs, a large, fast current is injected into the end of the axon. That causes the voltage to quickly spread into the axon – just as it did

in the dendrites (see Fig. 5b). Eventually the voltage at the next node of Ranvier is high enough to produce an action potential (see Fig. 5c), and the process repeats.

This is called saltatory conductance; the name comes from the fact that at the microscopic level, signal propagation is not smooth. Note that the signal can go in either direction. However, because sodium channels de-inactivate slowly (it takes a few ms for the h-current to decay to zero) and potassium channels inactivate slowly (it takes several ms for the n-current to decay), once saltatory conductance starts in one direction, it can't suddenly go in the other. And under normal conditions propagation is almost always away from the soma. However, electrophysiologists often deliberately cause propagation to go toward the soma; such propagation is referred to as antidromic.

The intuitive picture is the main thing we need to understand. However, as usual, we're going to do some math. We're going to solve the cable equation with a delta-function current, as in Eq. (46). The idea is that the nodes of Ranvier supply the delta-function, and voltage then diffuses along the myelinated part of the axon. However, we can't quite use Eq. (46) because it involves r_m, which is infinite for the myelinated part. We thus have to go back to Eq. (20). Because of the myelin there's no leak term (so $i_m = 0$). We'll divide both sides by c_m, put a delta-function on the right hand side, and not worry too much about units. The equation we want to solve is, then,

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + \delta(x)\delta(t) \tag{59}$$

where the diffusion constant, D, is given by

$$D \equiv \frac{a}{2r_L c_m}\,. \tag{60}$$

Note that r_L has units of resistance times length (Eq. (18)) and c_m has units of capacitance per area (Eq. (19)), so D has units of length^2/time. Here u is membrane potential relative to the resting membrane potential at the nodes of Ranvier.

We'll use L to denote the spacing between the nodes of Ranvier. Given our above discussion of saltatory conductance, the relevant quantity is the membrane potential at the next node, at $x = L$. That's given by.

$$u(L,t) = \frac{e^{-L^2/4Dt}}{\sqrt{4\pi Dt}}\Theta(t)\,. \tag{61}$$

The form of this equation follows from Eq. (47) and two observations: $D = \lambda^2/\tau_m$, a quantity that is independent of r_m, and when $r_m \to \infty$, τ_m also goes to ∞. To see that the amplitude is correct, note that

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t) = \delta(t), \tag{62}$$

which implies that when $t > 0$, $u(x,t)$ must integrate to 1 – as it does in Eq. (61).

We're interested in the maximum value of $u(L,t)$ as a function of t, which we denote $u(L,t^*)$. This we can find by setting the derivative of the right hand side of Eq. (61) with respect to t to zero. After a small amount of algebra, we find that $t^* = L^2/2D$, yielding

$$u(L,t^*) = \frac{1}{\sqrt{2\pi e}}\,\frac{1}{L}\,. \tag{63}$$

This indicates that the maximum membrane potential at the next node of Ranvier – which needs to be above threshold for the generation of an action potential – falls off rather slowly with distance. This slow falloff is also apparent in plots of $u(L,t)$ versus t for various values of L (see Fig. 5c). The slow falloff of the maximum voltage doesn't put a strong constraint on L. For those who are interested, there's a good discussion of the actual constraints – which involve propagation speed – in Dayan and Abbott.

6 Synapses

The conductances can also depend on the concentration of a neurotransmitter in the synaptic cleft (the area between the red presynaptic terminal and the green spine in Fig. 1). In that case the channels are on the spine, and we have

$$I_s = g_s(V - \mathcal{E}_s) \tag{64a}$$

$$g_s = \overline{g}_s s \tag{64b}$$

where s (which stands for "synaptic"), is between 0 and 1. It obeys the equation

$$\frac{ds}{dt} = c(1-s) - \beta s\,. \tag{65}$$

Here c is the neurotransmitter concentration in the synaptic cleft (which is generally near zero, but goes up when a spike arrives at the presynaptic terminal), and β tells us how fast the synaptic conductance decays when the concentration drops back to near zero.

There is a bit of a subtlety associated with Eq. (64). The voltage should really refer to the voltage in the spine, not at the soma. However, to model networks, we often pretend that it's the voltage at the soma; basically, we pretend that dendrites don't exist (this is the point neuron approximation). In that case, the current, $I_s = s\overline{g}_s(V - \mathcal{E}_s)$, is the current that flows into the soma.

7 A model of networks of neurons, on fast timescales

With this approximation, we can combine Eq. (64) with the Hodgkin-Huxley equation, Eq. (10), to give us a set of equations describing a network of neurons. Using the subscript i to label neurons, and letting $\mathcal{E}_s \to \mathcal{E}_j$, $s(t) \to s_j(t)$ and $\overline{g}_s \to W_{ij}$ (and summing over j), we have a set of equations that looks like

$$C\,\frac{dV_i}{dt} = -g_L(V_i - \mathcal{E}_L) - g_{Na} m_i^3 h_i (V_i - \mathcal{E}_{Na}) - g_K n_i^4 (V_i - \mathcal{E}_K) - \sum_j W_{ij} s_j(t)(V_i - \mathcal{E}_j)\,. \tag{66}$$

We say "on fast timescales" because it ignores the fact that the weights change, and weight changes depend on activity.

There are several things to note about this equation. First, the reversal potential, $\mathcal{E}_j$, depends on the presynaptic neuron – something that evolution gave us. Second, we should be aware that the weights, W_{ij}, are very sparse: each neuron makes only about 1,000 connections, and a brain the size of, say, a human, contains 100 billion neurons, so most of

the weights are zero. Third $s_j(t)$ determines the shape of the PSP (post-synaptic potential) associated with neuron j. It obeys something like Eq. (65), but we often assume it has a stereotyped shape, and write

$$s_j(t) = \sum_k f_j(t - t_j^k) \tag{67}$$

where t_j^k is the time of the k^{th} spike on neuron j and $f_j(t)$ is a function that rises rapidly and decays slightly more slowly than it rises. It is sometimes modeled as a double exponential,

$$f_j(t) = \frac{e^{-t/\tau_j^s} - e^{-t/\tau_j^f}}{\tau_j^s - \tau_j^f} \Theta(t)\,. \tag{68}$$

Here τ_j^s and τ_s^f are fast and slow time constants; for fast synapses, τ_f^f is in the range 1-5 ms and τ_j^s is in the range 3-10 ms (and they can be many 10s of ms for slow synapses, or synapses far from the soma), and $\Theta(t)$ is the usual Heaviside step function (defined above). However, we could swap in just about any function and that wouldn't have much effect on the network dynamics.

8 Summary

As you can see, things are relatively complicated. But just keep in mind two things:

1. All of biophysics comes from Eq. (5).

2. Conductances, g_x, are the interesting part of Eq. (5). So far we have seen that they can depend on voltage and the concentration of a neurotransmitter. (They can, of course, depend on both – this being biology, evolution has thought of just about anything we can imagine, within reason.) But that's not all. For instance, for very early sensory processing, conductances can depend on the outside world: photoreceptors in the retina have conductances that respond to light; hair cells in the ear have conductances that respond to mechanical vibration; the olfactory receptor neurons in the nose have conductances that respond to chemicals, and so on. So, if we want to know the fundamental equations describing the brain, we need to focus on conductances!

Finally, we have not talked about synaptic plasticity. That, in my opinion, is the glaring hole in our knowledge of the brain: we don't really know what the learning rules are. Maybe in a decade or two we will, and I can update the notes.